Ex 1: Prove $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

Proof: Note that $\forall\, x, u \in \mathbb{R}$,

$$\left|f(x) - f(u)\right| = \left|\frac{1}{1+x^2} - \frac{1}{1+u^2}\right| = \underbrace{\frac{|x+u|}{(1+x^2)(1+u^2)}}_{\text{need to control this.}}\,|x-u|.$$

We have

$$\frac{|x+u|}{(1+x^2)(1+u^2)} \le \frac{|x|+|u|}{(1+x^2)(1+u^2)} \le \frac{|x|}{1+x^2} + \frac{|u|}{1+u^2} \le \frac{1}{2} + \frac{1}{2} = 1.$$

Let $\varepsilon > 0$ be arbitrary. Take $\delta = \varepsilon$.

Then whenever $x, u \in \mathbb{R}$ & $|x-u| < \delta$,

$$\left|f(x) - f(u)\right| \le |x-u| < \delta = \varepsilon.$$

By def, $f$ is uniformly continuous on $\mathbb{R}$. $\square$

Ex 2:

Prove that the function $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$.

Proof: Take $\varepsilon = 1$. For any $\delta > 0$, take

$$x = \frac{1}{\delta} + \frac{\delta}{2} \text{ and } u = \frac{1}{\delta}.$$

Then $|x - u| < \delta$ but

$$|f(x) - f(u)| = \left| \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2 \right| = 1 + \frac{\delta^2}{4} \geq 1.$$

$\square$

Nonuniform Continuity Criteria:

Let $A \subset \mathbb{R}$ & $f: A \to \mathbb{R}$. TFAE:

(i) $f$ is NOT uniformly continuous on $A$.

(ii) $\exists\, \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists\, x_\delta, u_\delta \in A$ s.t.

$$|x_\delta - u_\delta| < \delta \text{ and } |f(x_\delta) - f(u_\delta)| \geq \varepsilon_0.$$

(iii) $\exists\, \varepsilon_0 > 0$ & two sequences $(x_n)$, $(u_n)$ in $A$ s.t.

$$\lim (x_n - u_n) = 0 \text{ and } |f(x_n) - f(u_n)| \geq \varepsilon_0 \ \forall n.$$

Ex 3: Show that if $f$ is continuous on $[0,\infty)$ & uniformly continuous on $[a,\infty)$ for some $a>0$, then $f$ is uniformly continuous on $[0,\infty)$.

Proof: Let $\varepsilon>0$.

Since $f$ is uniformly continuous on $[a,\infty)$,

$\exists\, \delta_1>0$ s.t. if $x,u\in[a,\infty)$ & $|x-u|<\delta_1$,

then $|f(x)-f(u)|<\varepsilon$.

By Uniform Continuity Theorem,

$f$ is uniformly continuous on $[0,a+\delta_1]$.

Hence, $\exists\, \delta_2>0$ s.t. if $x,u\in[0,a+\delta_1]$ & $|x-u|<\delta_2$,

then $|f(x)-f(u)|<\varepsilon$.

Take $\delta=\min\{\delta_1,\delta_2\}$. Then whenever $x,u\in[0,\infty)$ & $|x-u|<\delta$,

either $x,u\in[0,a+\delta_1]$ or $x,u\in[a,\infty)$.

Hence in both cases, we have

$|f(x)-f(u)|<\varepsilon$.

Thus $f$ is uniformly continuous on $[0,\infty)$. ∎